Reduced-order modeling of transport equations using Wasserstein spaces

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Outline of the talk

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Introduction to Wassertein spaces and barycenters

Model order reduction of parametric transport equations

Probability measures with finite second-order moments

Let $d \in \mathbb{N}^*$ and $\Omega \subset \mathbb{R}^d$ an open domain.

Let $\mathcal{P}_2(\Omega)$ denote the set of **probability measures** u on Ω with finite second-order moments, i.e.

$$\int_\Omega u(dx) = 1, \quad \int_\Omega (1 + |x|)^2 \, u(dx) < +\infty.$$

Example: Let $\rho \in L^1(\Omega)$ such that

$$\rho \geq 0, \quad \int_\Omega \rho(x) \, dx = 1, \quad \int_\Omega (1 + |x|)^2 \rho(x) \, dx < +\infty. \tag{1}$$

Then, the probability measure $u(dx) := \rho(x) \, dx$ belongs to $\mathcal{P}_2(\Omega)$.

Wasserstein space

The 2-**Wasserstein (or Kantorovich-Rubinstein) metric** is a distance function defined between two probability measures $u_1, u_2 \in \mathcal{P}_2(\Omega)$ and is denoted by

$$W_2(u_1, u_2).$$

The set $(\mathcal{P}_2(\Omega), W_2)$ then defines a metric space, called the **Wasserstein space**.

Its precise definition will come later... Patience!

I first would like to explain to you some interesting properties of this distance with respect to **model-reduction of parametric transport dominated problems.**

Interpolation in the Wasserstein or $L^2(\Omega)$ space

Let $\rho_1, \rho_2 \in L^2(\Omega) \cap L^1(\Omega)$ which satisfy (1). Define $u_1, u_2 \in \mathcal{P}_2(\Omega)$ such that $u_i(\,dx) = \rho_i(x)\,dx$ for $i = 1, 2$.

Let $t \in [0, 1]$ and consider the two following problems:

- **Interpolation in the $L^2(\Omega)$ space:** Find $\rho_t^{L^2} \in L^2(\Omega)$ such that

$$\rho_t^{L^2} = \underset{\rho \in L^2(\Omega)}{\operatorname{argmin}} (1-t)\|\rho - \rho_1\|^2_{L^2(\Omega)} + t\|\rho - \rho_2\|^2_{L^2(\Omega)}.$$

  Then, we all know that the solution is $\rho_t^{L^2}$ is the barycentric combination of ρ_1 and ρ_2, i.e.

$$\rho_t^{L^2} := (1-t)\rho_1 + t\rho_2.$$

- **Interpolation in the Wasserstein space:** Find $u_t \in \mathcal{P}_2(\Omega)$ such that

$$u_t = \underset{u \in \mathcal{P}_2(\Omega)}{\operatorname{argmin}} (1-t) W_2(u, u_1)^2 + t W_2(u, u_2)^2.$$

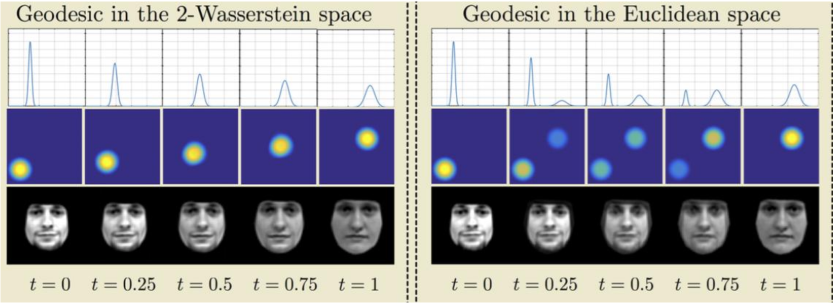
  The measure u_t is unique and is called the **McCann's interpolant** between u_1 and u_2. It holds that $u_t(\,dx) = \rho_t^{W_2}(x)\,dx$ for some $\rho_t^{W_2} \in L^1(\Omega)$.

What does $\rho_t^{L^2}$ and $\rho_t^{W_2}$ look like?

Comparison between the Wasserstein and $L^2(\Omega)$ interpolation

[Kolouri et al. 2016]

$\rho_t^{W_2}$ $\rho_t^{L^2}$



Interesting property of the Wasserstein metric for transport-dominated problems:

If $\rho_2 = \rho_1(\cdot - c)$ for some $c \in \mathbb{R}^d$, then it holds that

$$\rho_t^{W_2} = \rho_1(\cdot - tc).$$

Barycenters in the Wasserstein or $L^2(\Omega)$ space

Let $n \in \mathbb{N}^*$ and $U := (u_1, u_2, \cdots, u_n) \in \mathcal{P}_2(\Omega)^n$. Let $\Lambda := (\lambda_1, \cdots, \lambda_n) \in [0, 1]^n$ such that $\sum_{i=1}^n \lambda_i = 1$, and consider the minimization problem:

Find $\mathrm{Bar}(U, \Lambda) \in \mathcal{P}_2(\Omega)$ such that

$$\mathrm{Bar}(U, \Lambda) = \underset{u \in \mathcal{P}_2(\Omega)}{\mathrm{argmin}} \sum_{i=1}^n \lambda_i W_2(u, u_i)^2.$$

The measure $\mathrm{Bar}(U, \Lambda)$ is unique and is called the **Wasserstein barycenter** of U with weights Λ.

This object is the Wasserstein counterpart of the $L^2(\Omega)$ barycenter of a set of functions $(\rho_1, \cdots, \rho_n) \in L^2(\Omega)^n$ with barycentric weight Λ. Indeed,

$$\rho_\Lambda^{L^2} := \sum_{i=1}^n \lambda_i \rho_i,$$

is equivalently the unique minimizer of

$$\rho_\Lambda^{L^2} = \underset{\rho \in L^2(\Omega)}{\mathrm{argmin}} \sum_{i=1}^n \lambda_i \|\rho - \rho_i\|_{L^2(\Omega)}^2.$$

Wasserstein barycenters

Barycenters in the Wasserstein space

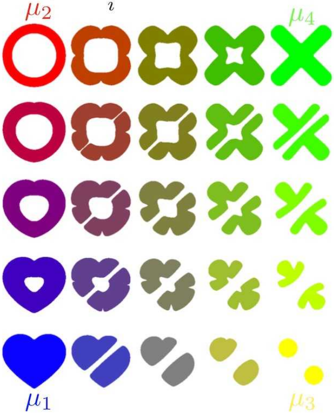
Martial Agueh *, Guillaume Carlier †

August 17, 2010

Abstract

In this paper, we introduce a notion of barycenter in the Wasserstein space which generalizes McCann's interpolation to the case of more than two measures. We provide existence, uniqueness, characterizations and regularity of the barycenter, and relate it to the multi-marginal optimal transport problem considered by Gangbo and Święch in [8]. We also consider some examples and in particular rigorously solve the gaussian case. We finally discuss convexity of functionals in the Wasserstein space.

$$\min_{\mu} \sum_i \lambda_i W_2^2(\mu_i, \mu)$$



Wasserstein metric: optimal transport

Let $u_1, u_2 \in \mathcal{P}_2(\Omega)$. Then,

$$W_2(u_1, u_2)^2 := \inf_{\substack{\pi \in \mathcal{P}(\Omega \times \Omega) \\ \int_{y \in \Omega} \pi(dx, dy) = u_1(dx) \\ \int_{x \in \Omega} \pi(dx, dy) = u_2(dy)}} \int_{\Omega \times \Omega} |x - y|^2 \, \pi(dx, dy).$$

where $\mathcal{P}(\Omega \times \Omega)$ is the set of probability measures on $\Omega \times \Omega$.

Kantorovich formulation of optimal transport problem

Several numerical methods exist for solving such problems and computing Wasserstein barycenters: linear programming, auction algorithm, entropic regularization...

Special case of dimension 1

Let us focus on the special case where $d = 1$, where Wasserstein distance and Wasserstein barycenters can be computed explicitly. From now on, let us now assume that $\Omega \subset \mathbb{R}$ is an interval of $\mathbb{R}$.

Let $u \in \mathcal{P}_2(\Omega)$. Let $\mathrm{cdf}_u : \Omega \to [0, 1]$ be the **cumulative distribution of** u, defined by:

$$\forall x \in \Omega, \quad \mathrm{cdf}_u(x) := \int_{-\infty}^{x} u(dy),$$

Let $\mathrm{icdf}_u : [0, 1] \to \Omega \subset \mathbb{R}$ be the **inverse cumulative distribution of** u, the generalized inverse of cdf_u, defined as follows:

$$\forall s \in [0, 1], \quad \mathrm{icdf}_u(s) := \inf \{x \in \Omega, \quad \mathrm{cdf}_u(x) > s\} .$$

Let us denote by $\mathcal{I} := \{\mathrm{icdf}_u, \ u \in \mathcal{P}_2(\Omega)\}$. Then, it holds that $\mathcal{I}$ is a convex subset of $L^2(0, 1)$.

Wasserstein distance and barycenter in dimension 1

For all $u, v \in \mathcal{P}_2(\Omega)$, the 2-Wasserstein distance between u and v is equal to

$$W_2(u, v) := \|\mathrm{icdf}_u - \mathrm{icdf}_v\|_{L^2(0,1)}.$$

Let $U := (u_1, \cdots, u_n) \in \mathcal{P}_2(\Omega)^n$ and $\Lambda := (\lambda_1, \cdots, \lambda_n) \in [0, 1]^n$ such that $\sum_{i=1}^n \lambda_i = 1$.

The Wasserstein barycenter of the family U with barycentric weights Λ is the unique measure $\mathrm{Bar}(U, \Lambda) \in \mathcal{P}_2(\Omega)$ such that

$$\mathrm{icdf}_{\mathrm{Bar}(U,\Lambda)} = \sum_{i=1}^n \lambda_i \, \mathrm{icdf}_{u_i},$$

Outline of the talk

Parametric conservative transport equations

Consider a parametric conservative transport equation of the form: $t \in [0, T]$ with $T > 0$, $x \in \Omega \subset \mathbb{R}$.

$$\begin{cases} \partial_t \rho_\mu(t,x) - \partial_x F(\mu, \rho_\mu(t,x)) = 0, \\ \rho_\mu(t=0,x) = \rho_{\mu,0}(x), \\ +\text{boundary conditions}, \end{cases} \tag{2}$$

where μ is a vector of parameters belonging to a set $\mathcal{P} \subset \mathbb{R}^p$.

Assumption: Assume that for all $\mu \in \mathcal{P}$, $t \in [0, T]$,

$$\rho_\mu(t,\cdot) \geq 0, \quad \int_{\mathbb{R}} \rho_\mu(t,\cdot) = 1 \quad \text{and} \int_{\mathbb{R}} (1+|x|)^2 \rho_\mu(t,x)\, dx < +\infty.$$

Then, let $u_\mu(t)(\,dx) = \rho_\mu(t,x)\, dx$ the associated probability measure so that $u_\mu(t) \in \mathcal{P}_2(\Omega)$.

Example: pure transport one-dimensional equation

Consider the following simple pure one-dimensional transport equation:
$t \in [0, T]$ with $T > 0$, $x \in \Omega := \mathbb{R}$.

$$\begin{cases} \partial_t \rho_\mu(t, x) - \mu \partial_x \rho_\mu(t, x) = 0, \\ \rho_\mu(t = 0, x) = \rho_0(x), \end{cases} \tag{3}$$

where $\mu \in \mathcal{P} := [\mu_{\min}, \mu_{\max}] \subset \mathbb{R}$ and $u_0(\,dx) := \rho_0(x)\,dx \in \mathcal{P}_2(\Omega)$.

The solution to (5) is well-known to be

$$\rho_\mu(t, x) := \rho_0(x + \mu t), .$$

Solution set

Problem: Build a reduced-order model approximation of the solution set

$$\mathcal{M} := \{\rho_\mu(t,\cdot),\ \mu \in \mathcal{P},\ t \in [0,T]\}$$

Assume for the sake of simplicity that $\mathcal{M} \subset L^2(\Omega)$.

Linear approximation methods

Assume that $\mathcal{M} \subset V$ where V is a Hilbert space (for instance $V = L^2(\Omega)$).

A classical method to construct reduced-order models consists in looking for a n-dimensional linear subspace V_n of V (with n small) so that the error

$$\mathrm{dist}^V(\mathcal{M}, V_n) := \sup_{u \in \mathcal{M}} \|u - \Pi_{V_n} u\|_V$$

is as small as possible (here Π_{V_n} denotes the orthogonal projection of V onto V_n).

For a fixed value of n, the **best possible approximation error** is given by the **Kolmogorov n-width** of the set $\mathcal{M}$, defined as

$$d_n^V(\mathcal{M}) := \inf_{\substack{V_n \subset V, \\ \dim V_n = n}} \mathrm{dist}^V(\mathcal{M}, V_n).$$

Greedy algorithms used in reduced basis methods provides a practical way to find a quasi-optimal linear subspace V_n in many situations.

[DeVore et al., 2013]

Slow decay of the Kolmogorov n-width for hyperbolic equations

For elliptic and parabolic equations, $\left(d_n^V(\mathcal{M})\right)_{n\in\mathbb{N}^*}$ decays fast as n goes to infinity. [Cohen et al., 2016]

Problem! For transport equations, $\left(d_n^V(\mathcal{M})\right)_{n\in\mathbb{N}^*}$ may decay quite slowly as n goes to infinity.

Example: one-dimensional pure transport equation [Ohlberger, Rave, 2016]. There exists $c > 0$ such that for all $n \ni \mathbb{N}^*$,

$$\boxed{d_n^{L^2(\Omega)}(\mathcal{M}) \geq cn^{-1/2}}$$

Nonlinear approximation methods have to be used!

Non-exhaustive list of related works:

Billaud-Friess, Cagniart, Carlberg, Falco, Guignard, Maday, Mehrmann, Musharbash, Nobile, Pagliantini, Peherstorfer, Stamm, Welper, Willcox...

Yet another possible viewpoint... with Wasserstein!

New solution set: Let

$$\widetilde{\mathcal{M}} := \{u_\mu(t),\ \mu \in \mathcal{P},\ t \in [0,T]\} \subset \mathcal{P}_2(\Omega),$$

and, motivated by the properties of the Wasserstein metric and its expression for one-dimensional problems, let us rather consider the set composed of the inverse cumulative distribution functions for all $u \in \widetilde{\mathcal{M}}$

$$\mathcal{T} := \left\{ \mathrm{icdf}_u,\ u \in \widetilde{\mathcal{M}} \right\} \subset \mathcal{I} \subset L^2(0,1)$$

Let's try to reduce the transformed set $\mathcal{T}$ in $L^2(0,1)$ rather than the original solution set $\mathcal{M}$ in $L^2(\Omega)$!

Wasserstein: case of the pure transport equation

$$\begin{cases} \partial_t \rho_\mu(t,x) - \mu \partial_x \rho_\mu(t,x) = 0, \\ \rho_\mu(t=0,x) = \rho_0(x), \end{cases} \tag{4}$$

test test2 test3

Figure: Pure transport equation

It holds that

$$\forall s \in (0,1), \quad \mathrm{icdf}_{u_\mu(t)}(s) := \mathrm{icdf}_{u_0}(s) - \mu t.$$

As a consequence,

$$d_n^{L^2(0,1)}(\mathcal{T}) = 0, \quad \forall n \geq 2.$$

Case of inviscid Burgers equation

For $t \in [0, T]$ with $T = 5$, $x \in \Omega := (-1, 4)$.

$$\begin{cases} \partial_t \rho_\mu(t,x) - \frac{1}{2}\partial_x \left({\rho_\mu}^2\right)(t,x) = 0, \\ \rho_\mu(t=0,x) = \rho_{\mu,0}(x), \end{cases} \tag{5}$$

with

$$\rho_{\mu,0}(x) := \begin{cases} 0 & \text{if } -1 \leq x < 0, \\ \mu & \text{if } 0 \leq x < \frac{1}{\mu}, \\ 0 & \text{if } \frac{1}{\mu} \leq x \leq 4. \end{cases} \qquad \text{where } \mu \in \mathcal{P} := [0.5, 3].$$

test test2 test3

Figure: Inviscid Burgers equation ($\mu = 1$)

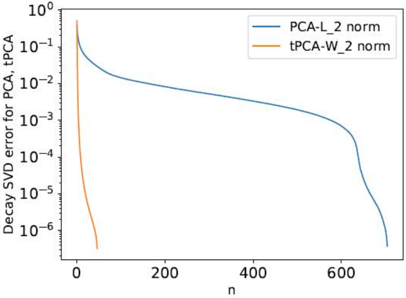
Case of inviscid Burgers equation

Proposition (VE,Lombardi,Mula, Vialard, 2020)

There exsits $C > 0$ *such that for all* $n \in \mathbb{N}^*$,

$$\boxed{d_n^{L^2(0,1)}(\mathcal{T}) \leq Cn^{-21/10}}$$

No proof of lower bounds on $d_n^{L^2(\Omega)}(\mathcal{M})$, but numerical results that seem to indicate that the latter may decay slower as n goes to infinity than $d_n^{L^2(0,1)}(\mathcal{T})$.



What next?

This suggests the development of algorithms which

- exploit this structure and the properties of the Wasserstein metric;
- can be transposed to problems in dimension $d > 1$;

in order to construct reduced-order models.

Computation of a selection of snapshots in the offline phase : Let $N \in \mathbb{N}^*$ (possibly large), choose $\mu_1, \cdots, \mu_N \in \mathcal{P}$ and $t_1, \cdots, t_N \in [0, T]$, and define the trial set of N snapshots. For all $1 \leq i \leq N$, denote by $u_i := u_{\mu_i}(t_i)$ and by

$$\widetilde{\mathcal{M}}_{\rm trial} := \{u_{\mu_i}(t_i),\ i = 1, \cdots, N\} = \{u_i,\ i = 1, \cdots, N\}.$$

Tangent PCA algorithm: offline phase

The first method we consider is the so-called **tangent PCA**, and is based on a POD method.

Define $\overline{u} \in \mathcal{P}_2(\Omega)$ by

$$\overline{u} = \frac{1}{N} \sum_{i=1}^{N} u_i$$

and consider the set of functions

$$\mathrm{icdf}_{u_i} - \mathrm{icdf}_{\overline{u}} \in L^2(0,1), \quad i = 1, \cdots, N.$$

Compute the n first PCA/POD modes of this set of functions, denoted by $f_1, \cdots, f_n \in L^2(0,1)$ and for all $i = 1, \cdots, N$, compute

$$c_k^i := \langle \mathrm{icdf}_{u_i} - \mathrm{icdf}_{\overline{u}}, f_k \rangle_{L^2(0,1)}, \ \forall k = 1, \cdots, n,$$

so that $\sum_{k=1}^{n} c_k^i f_k = \Pi_{\mathrm{Span}\{f_1, \cdots, f_n\}} \left(\mathrm{icdf}_{u_i} - \mathrm{icdf}_{\overline{u}} \right)$ and

$$\mathrm{icdf}_{u_i} \approx \mathrm{icdf}_{\overline{u}} + \sum_{k=1}^{n} c_k^i f_k = \mathrm{icdf}_{\overline{u}} + \Pi_{\mathrm{Span}\{f_1, \cdots, f_n\}} \left(\mathrm{icdf}_{u_i} - \mathrm{icdf}_{\overline{u}} \right).$$

Store $(f_k)_{1 \le k \le n}$ and $(c_k^i)_{1 \le k \le n, 1 \le i \le N}$.

tangent PCA algorithm: online phase

For all $1 \leq k \leq n$, let us consider an interpolation $\overline{c}_k : \mathcal{P} \times [0, T] \to \mathbb{R}$ so that for all $1 \leq i \leq N$, $\overline{c}_k(\mu_i, t_i) = c_k^i$.

For $\mu \in \mathcal{P}$, $t \in [0, T]$, approximate $u_\mu(t)$ by $u_\mu^n(t)$ defined so that

$$\mathrm{icdf}_{u_\mu^n(t)} = \mathrm{icdf}_{\overline{u}} + \sum_{k=1}^{n} \overline{c}_k(\mu, t) f_k. \tag{6}$$

tangent PCA algorithm: pros and cons

Advantages:

- optimality of the PCA!
- extendable to higher dimension, exploiting the fact that the Wasserstein space has the structure of a Riemannian manifold.

Problem: Such a method is not robust. Indeed, all functions $f_k \in L^2(0,1)$ but there is no guarantee that

$$\mathrm{icdf}_{\overline{u}} + \sum_{k=1}^{n} \overline{c}_k(\mu, t) f_k$$

belongs to $\mathcal{I} := \{\mathrm{icdf}_u,\ u \in \mathcal{P}_2(\Omega)\}$. In other words, $u_\mu^n(t)$ may not be well-defined via formula (6).

Such a problem can in principle be solved (even in higher dimension) using the so-called **geodesic PCA**[Bigot et al., 2017]. However, computing the geodesic PCA is very expensive from a computational point of view.

Need for a more robust and cheap numerical strategy...

Barycentric Greedy algorithm: main ingredients

The second method we consider is based on a **greedy algorithm** and on the fact that

$$\mathcal{I} := \{\mathrm{icdf}_u,\ u \in \mathcal{P}_2(\mathbb{R})\}$$

is a **closed convex subset of $L^2(0,1)$**.

For all $n \in \mathbb{N}^*$, define the set of barycentric weights

$$\Sigma_n := \left\{ \Lambda := (\lambda_1, \cdots, \lambda_n) \in [0,1]^n,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Barycentric Greedy algorithm: offline phase

The **Barycentric greedy algorithm** is an iterative algorithm which reads as follows:

- **Initialization:** Choose $1 \leq i_1, i_2 \leq N$ such that

$$(i_1, i_2) \in \operatorname*{argmax}_{1 \leq i,j \leq N} W_2(u_i, u_j)^2.$$

  Define $U_2 := (u_{i_1}, u_{i_2})$ and set $n = 2$.

- **Iteration** $n \geq 2$**:**
  Choose $1 \leq i_{n+1} \leq N$ such that

$$i_{n+1} \in \operatorname*{argmax}_{1 \leq i \leq N} \min_{\Lambda \in \Sigma_n} W_2 \left(u_i, \mathrm{Bar}(U_n, \Lambda)\right)^2$$

  where $\mathrm{Bar}(U_n, \Lambda)$ denotes the Wasserstein barycenter of the family U_n with barycentric weights Λ.
  Define $U_{n+1} := (u_{i_1}, u_{i_2}, \cdots, u_{i_{n+1}})$ and set $n := n + 1$.

Barycentric Greedy algorithm: offline phase

The **Barycentric greedy algorithm** is an iterative algorithm which reads as follows:

- **Initialization:** Choose $1 \leq i_1, i_2 \leq N$ such that

  $$(i_1, i_2) \in \underset{1 \leq i,j \leq N}{\mathrm{argmax}} \left\| \mathrm{icdf}_{u_i} - \mathrm{icdf}_{u_j} \right\|^2_{L^2(0,1)}.$$

  Define $U_2 := (u_{i_1}, u_{i_2})$ and set $n = 2$.

- **Iteration $n \geq 2$:**
  Choose $1 \leq i_{n+1} \leq N$ such that

  $$i_{n+1} \in \underset{1 \leq i \leq N}{\mathrm{argmax}} \min_{\Lambda := (\lambda_k)_{1 \leq k \leq n} \in \Sigma_n} \left\| \mathrm{icdf}_{u_i} - \sum_{k=1}^{n} \lambda_k \mathrm{icdf}_{u_{i_k}} \right\|^2_{L^2(0,1)}$$

  Define $U_{n+1} := (u_{i_1}, u_{i_2}, \cdots, u_{i_{n+1}})$ and set $n := n + 1$.

Each step of the greedy algorithm only requires the minimization of a quadratic functional on a convex set defined by affine constraints.

Barycentric Greedy algorithm: offline phase

For fixed $n \in \mathbb{N}^*$, store $U_n := (u_{i_1}, \cdots, u_{i_n})$ and for all $i = 1, \cdots, N$, store $\Lambda^i := (\lambda^i_k)_{1 \le k \le n} \in \Sigma_n$ such that

$$\Lambda^i \in \operatorname*{argmin}_{\Lambda \in \Sigma_n} W_2 \left(u_i, \mathrm{Bar}(U_n, \Lambda)\right)^2 .$$

Let $\overline{\Lambda} : \mathcal{P} \times [0, T] \to \Sigma_n$ be an interpolation, i.e. a function such that for all $1 \le i \le N$,

$$\overline{\Lambda}(\mu_i, t_i) = \Lambda^i.$$

Barycentric Greedy algorithm: online phase

For given $\mu \in \mathcal{P}$ and $t \in [0, T]$, compute $u_\mu^n(t) \in \mathcal{P}_2(\Omega)$ as the Wasserstein barycenter of the family U_n with barycentric weights $\overline{\Lambda}(\mu, t)$, i.e.

$$u_\mu^n(t) = \mathrm{Bar}(U_n, \overline{\Lambda}(\mu, t)).$$

This amounts to computing

$$\mathrm{icdf}_{u_\mu^n(t)} = \sum_{k=1}^{n} \overline{\lambda}_k(\mu, t)\mathrm{icdf}_{u_{i_k}}.$$

Barycentric Greedy algorithms: pros and cons

Drawback: No optimality or quasi-optimality guarantee of such a barycentric greedy procedure.

Advantages:

- The method is robust since $\mathcal{I}$ is a closed convex subset of $L^2(0,1)$;
- It can be extended to higher dimension $d > 1$;
- It can be generalised to other metric spaces.

Numerical s

Inviscid Burger's equation

$$\partial_t \rho_\mu + \frac{1}{2}\partial_x(\rho_\mu^2) = 0$$

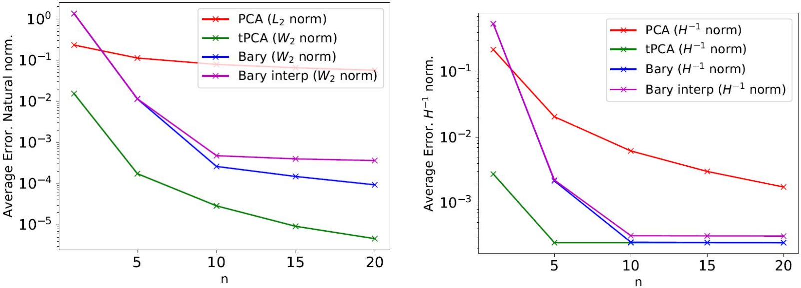


Figure: Errors in natural norms (left) and H^{-1} norm (right) (average over the set of parameters)

Numerical tests

Inviscid Burger's equation

$$\partial_t \rho_\mu + \frac{1}{2}\partial_x(\rho_\mu^2) = 0$$

test

Figure: Inviscid Burgers equation

Numerical tests

Viscous Burger's equation

$$\partial_t \rho_\mu + \frac{1}{2}\partial_x(\rho_\mu^2) - \nu \partial_x^2 \rho_\mu = 0$$

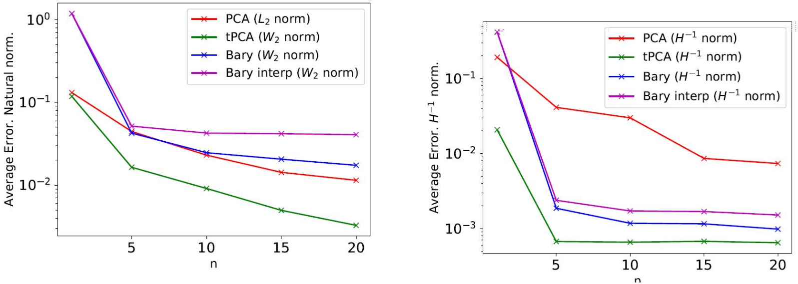


Figure: Errors in natural norms (left) and H^{-1} norm (right) (average over the set of parameters)

Numerical tests

Viscous Burger's equation

$$\partial_t \rho_{\mu,\nu} + \frac{1}{2}\partial_x(\rho_{\mu,\nu}^2) - \nu\partial_x^2\rho_{\mu,\nu} = 0$$

test

Figure: Viscous Burgers equation

Conclusions

- Nonlinear approximation methods in the Wasserstein space (which is a metric space and not a vectorial spaces) for the reduction of transport-dominated problems.
- Theoretical result on the decay of the Kolmogorov n-width of the transformed set of solutions.
- Two numerical methods: (i) tangent PCA method (ii) Wasserstein barycenters together with a barycentric greedy algorithm.
- Advantages: the methods are non-intrusive, and enjoy the nice interpolation properties of the Wasserstein metric with respect to transport-dominated problems.

Persectives

- Extension to higher dimension and non-conservative transport equations.
- Acceleration of the offline phase.
- A posteriori error estimators.
- How can we introduce back the PDE in the approximation scheme?
- Such an approach can be extended to general metric spaces. How to choose (and compute) the *best* (or a quasi-best) metric space associated to a solution set? Can we learn the metric?

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